JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 55, 490–508 (1976)

Extensions of Two Person Zero Sum Games*

HERVÉ MOULIN

143 B^{d}Lefebvre, 75015 Paris, France

Submitted by J. L. Lions

INTRODUCTION

A two person zero sum game describes a world that the two players influence and appreciate in opposite ways: One's gain is always equal to the other's loss. We call these two players Xavier and Yves. We denote by X the set of pure strategies of Xavier and by Y the set of pure strategies of Yves. If Xavier chooses a pure strategy $x \in X$ and Yves chooses a pure strategy $y \in Y$, then Xavier's payoff (his utility) is $g(x, y)$ where g is a function:

$$g: X \times Y \to \mathbb{R}. \tag{0.1}$$

Now, for the same pair (x, y), Yves's payoff is $-g(x, y)$. This means that Xavier tries to maximize g, while Yves tries to minimize it. There are basically two kinds of games.

Case 1. $\sup_x \inf_y g(x, y) = \inf_y \sup_x g(x, y)$.

Let us denote by v this common value and call it the *value* of the game. If we assume there is an $\bar{x}$ such that $\sup_x \inf_y g(x, y) = \inf_y g(\bar{x}, y)$, then Xavier, when playing $\bar{x}$, is guaranteed to have a payoff greater than or equal to v:

$$\forall y \in Y \qquad v \leqslant g(\bar{x}, y). \tag{0.2}$$

Similarly, if we assume there is a $\bar{y}$ such that $\inf_y \sup_x g(x, y) = \sup_x g(x, \bar{y})$, then Yves, when playing $\bar{y}$, is guaranteed to have a payoff greater than or equal to $-v$:

$$\forall x \in X\ g(x, \bar{y}) \leqslant v. \tag{0.3}$$

Thus, the pair $(\bar{x}, \bar{y})$ has the saddle point property:

$$\forall (x, y) \in X \times Y\ g(x, \bar{y}) \leqslant g(\bar{x}, \bar{y}) \leqslant g(\bar{x}, y). \tag{0.4}$$

* This is an abridged version of the author's Doctoral Dissertation. Easy proofs are omitted. For the difficult results the main steps of the proof are indicated.

This means that $\bar{x}$ is an optimal strategy of Xavier and $\bar{y}$ an optimal strategy of Yves.

Case 2. $\sup_x \inf_y g(x, y) < \inf_y \sup_x g(x, y)$.

In this case, there is no saddle point (0.4) and no optimal strategy for Xavier or Yves. We can just say that Xavier can enforce a payoff greater than or equal to $\sup_x \inf_y g(x, y)$ and that Yves can enforce a payoff less than or equal to $\inf_y \sup_x g(x, y)$. So, if we want to assign to a game a value in some sense, then this value must belong to the *duality interval*:

$$[\sup_x \inf_y g(x, y), \inf_y \sup_x g(x, y)]. \tag{0.5}$$

The aim of game theory is to provide, in this second case, some justification for the words "value" and "optimal strategies." For that purpose it will prove useful (or even necessary) to enrich the mathematical structure of the game.

One way (the usual way) is the "mixed extension" of the game (von Neumann–Morgenstern): We assume that Xavier can play not only a pure strategy $x \in X$, but also according to every probability distribution μ on X. Similarly, we assume that Yves can play every distribution of probability ν on Y. If the pair (μ, ν) is chosen by Xavier and Yves, the resulting payoff will be:

$$\int_{X\times Y} g(x, y)\, d\mu(x)\, d\nu(y). \tag{0.6}$$

The basic von Neumann minimax theorem asserts that this new game has always (for every initial payoff function g) a saddle point pair, and then a value, called the *mixed value* of the game (0.1) (in the case where X and Y are finite).

Another way is what we call here the "ergodic extension," which is described as follows: Xavier chooses a pure strategy x_1; then Yves, knowing x_1, chooses a pure strategy y_1; then Xavier, knowing x_1 and y_1, chooses an x_2; then Yves, knowing x_1, y_1 and x_2, chooses an y_2 and so on.... The new payoff is now

$$\lim_{N\to+\infty} \frac{1}{2N} \sum_{i=1}^{N} [g(x_i, y_i) + g(x_{i+1}, y_i)]. \tag{0.7}$$

We shall see (II, Section 3) that this new game has a saddle point pair (if, for instance, X and Y are finite), and the corresponding value is called the *ergodic value* of the game (0.1). (It is also shown that the ergodic value is the same if Yves is the "starting" player.)

In the two previous examples, we have defined two quite different "ways of playing" the initial game (0.1). In this paper we shall define a large family of abstract "extensions" (including the above mixed and ergodic extension). Then we shall classify these abstract extensions according to the type of exchange of information involved. Finally we shall give two characterizations of all possible "values" obtained by abstract extensions (including, therefore, the mixed value and the ergodic value).

Notation. If Z is a set, $\mathscr{A}(Z)$ denotes the vector space of bounded functions defined on Z. The space $\mathscr{A}(Z)$ is a Banach space, supplied with the sup norm:

$$h \in \mathscr{A}(Z) \, \| h \| = \sup_{z \in Z} | h(z) | . \tag{0.8}$$

The Banach space $\mathscr{A}(Z)$ is partially ordered by:

$$h_1 \geqslant h_2 : \forall z \in Z \; h_1(z) \geqslant h_2(z). \tag{0.9}$$

We denote by $\theta \in \mathscr{A}(Z)$ the constant function equal to 1. We denote by $\mathscr{A}'(Z)$ the topological dual of $\mathscr{A}(Z)$ and by $\mathscr{A}_1'(Z)$ the simplex of $\mathscr{A}'(Z)$, i.e., the convex (and weakly compact) subset of those linear forms $\mu \in \mathscr{A}'(Z)$ which are "positive" and have a "total weight" of 1:

$$\forall h \in \mathscr{A}(Z) \colon h \geqslant 0 \Rightarrow [\mu, h] \geqslant 0 \tag{0.10}$$

$$[\mu, \theta] = 1. \tag{0.11}$$

Finally we denote by δ the canonical one-to-one mapping:

$$\delta \colon Z \to \mathscr{A}_1'(Z). \; \forall z \in Z \; \forall h \in \mathscr{A}(Z) \colon [\delta_z , h] = h(z). \tag{0.12}$$

I. Extensions

I.1. *Definition*

Let X and Y be two given sets: We call X the set of pure strategies of Xavier and Y the set of pure strategies of Yves.

DEFINITION I.1. An extension p of the games on $X \times Y$ is a 5-tuple

$$p = (\mathscr{X}, \mathscr{Y}, i, j, \pi) \tag{I.1}$$

where i and j are two one-to-one mappings:

$$i : X \to \mathscr{X} \; j : Y \to \mathscr{Y} \tag{I.2}$$

where π is a linear operator

$$\pi : \mathscr{A}(X \times Y) \to \mathscr{A}(\mathscr{X} \times \mathscr{Y}) \tag{I.3}$$

such that

$$\pi \text{ is positive and } \pi(\theta) = \theta \tag{I.4}$$

and such that

$$\begin{aligned} &\text{(i)} \quad \forall g \in \mathscr{A}(X \times Y)\ \forall x \in X\ \inf_{y \in Y} g(x, y) \leqslant \inf_{\eta \in \mathscr{Y}} \pi g(i(x), \eta), \\ &\text{(ii)} \quad \forall g \in \mathscr{A}(X \times Y)\ \forall y \in Y\ \sup_{x \in X} g(x, y) \geqslant \sup_{\xi \in \mathscr{X}} \pi g(\xi, j(y)). \end{aligned} \tag{I.5}$$

We call $\mathscr{X}$ and $\mathscr{Y}$ the sets of extended strategies: The one-to-one mapping i identifies X with a subset of $\mathscr{X}$, and j identifies Y with a subset of $\mathscr{Y}$ (I.2). The choice of a function $g \in \mathscr{A}(X \times Y)$ defines an "initial game" (Xavier maximizing g and Yves minimizing g) and the "extended game" of this initial game is described in normal form by the payoff function:

$$\pi g : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}. \tag{I.6}$$

Thus we call π the extension operator. The assumption (I.4) means that π is increasing and π leaves the constant functions invariant. The assumption (I.5-i) means that, if Xavier plays the "pure" strategy $i(x)$ in the extended game, he guarantees at least the same payoff as guaranteed by playing the pure strategy x in the initial game. If (I.5-i) fails, Xavier could reasonably refuse to play the extended game. Similarly, the assumption (I.5-ii) means that using a "pure" strategy $j(y)$ in the extended game is not worse for Yves than using y in the initial game.

PROPOSITION I.1. *Let $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ be an extension of the games on $X \times Y$. We have:*

$$\forall g \in \mathscr{A}(X \times Y)\ \forall (x, y) \in X \times Y\ \pi g(i(x), j(y)) = g(x, y) \tag{I.7}$$

and

$$\begin{aligned} \forall g \in \mathscr{A}(X \times Y) \sup_x \inf_y g(x, y) &\leqslant \sup_\xi \inf_\eta \pi g(\xi, \eta) \\ &\leqslant \inf_\eta \sup_\xi \pi g(\xi, \eta) \leqslant \inf_y \sup_x g(x, y). \end{aligned} \tag{I.8}$$

Proof. Inequalities (I.8) follow clearly from (I.5) and (I.7) follows from Proposition I.2 below.

The property (I.7) shows that the restriction of πg to the subset $i(X) \times j(Y)$ of $\mathscr{X} \times \mathscr{Y}$ is equal to g, and justifies for πg the name: "extended payoff function." Moreover, the relation (I.8) shows that, through the extension, the duality interval must be nonincreasing. In other words, if the initial payoff function g has a value, the extended payoff function πg has the same value.

The most interesting extensions are those such that πg has always a value (even if g doesn't have any), and this value therefore belongs to the duality interval of g.

DEFINITION I.2. An extension $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ is said to be playable if:

$$\forall g \in \mathscr{A}(X \times Y) \sup_{\xi} \inf_{\eta} \pi g(\xi, \eta) = \inf_{\eta} \sup_{\xi} \pi g(\xi, \eta). \tag{I.9}$$

A playable extension describes then a "way of playing" the initial game (0.1), by extending the strategy spaces and the payoff function in such a way that the extended game is playable (has a value). In part III of this paper we give two different characterizations of the playable extensions.

I.2. *Adjoint Form of an Extension*

The adjoint form of an extension is the most suitable tool for its study. We denote by $\otimes$ the tensor product (which is unambiguously defined if μ or ν is a discrete measure):

$$\mu \in \mathscr{A}_1'(X); \ \nu \in \mathscr{A}_1'(Y) \ \mu \otimes \nu \in \mathscr{A}_1'(X \times Y). \tag{I.10}$$

PROPOSITION I.2. *Let X and Y be fixed. We assume that $(\mathscr{X}, \mathscr{Y})$ are fixed extended strategy spaces and (i, j) are one-to-one mappings satisfying* (I.2). *Then the relation*:

$$\forall g \in \mathscr{A}(X \times Y) \ \forall (\xi, \eta) \in \mathscr{X} \times \mathscr{Y} \ \pi g(\xi, \eta) = [\pi^*(\xi, \eta), g], \tag{I.11}$$

defines a bijective mapping, from the set of the linear operators π verifying (I.3), (I.4), (I.5), *onto the set of the mappings π^**:

$$\pi^* : \mathscr{X} \times \mathscr{Y} \to \mathscr{A}_1'(X \times Y), \tag{I.12}$$

verifying:

$$\begin{aligned} &\text{(i)} \quad \forall x \in X \quad \forall \eta \in \mathscr{Y} \quad \pi^*(i(x), \eta) \in \delta_x \otimes \mathscr{A}_1'(Y) \\ &\text{(ii)} \quad \forall y \in Y \quad \forall \xi \in \mathscr{X} \quad \pi^*(\xi, j(y)) \in \mathscr{A}_1'(X) \otimes \delta_y . \end{aligned} \tag{I.13}$$

Proof. The definition (I.11) of π^* can be rewritten as:

$$\forall g \in \mathscr{A}(X \times Y) \ \forall (\xi, \eta) \in \mathscr{X} \times \mathscr{Y} \ [\delta_\xi \otimes \delta_\eta , \pi g] = [\pi^*(\xi, \eta), g] \tag{I.14}$$

where the first bracket corresponds to the duality between $\mathscr{A}'(\mathscr{X} \times \mathscr{Y})$ and $\mathscr{A}(\mathscr{X} \times \mathscr{Y})$, and the second bracket corresponds to the duality between $\mathscr{A}'(X \times Y)$ and $\mathscr{A}(X \times Y)$. Thus π^* appears as the restriction to $\delta(\mathscr{X}) \otimes \delta(\mathscr{Y})$ of the adjoint operator of π. The end of the proof is a standard exercise on adjoint operators, using the Hahn–Banach theorem.

I.3. *Examples*

(a) The main example of extension of the games on $X \times Y$ is the *mixed extension* p_m :

$$p_m = (\mathscr{A}_1'(X), \mathscr{A}_1'(Y), \delta_X, \delta_Y, \pi_m), \tag{I.15}$$

where δ_X and δ_Y are the canonical embeddings (0.12), identifying X and Y with the Dirac measures of $\mathscr{A}_1'(X)$ and $\mathscr{A}_1'(Y)$, and where π_m is the following operator (*we assume that X and/or Y is finite*):

$$\forall g \in \mathscr{A}(X \times Y)\ \forall(\mu, \nu) \in \mathscr{A}_1'(X) \times \mathscr{A}_1'(Y) : \pi_m g(\mu, \nu) = [\mu \otimes \nu, g]. \tag{I.16}$$

In other words, the adjoint form $\pi_m{}^*$ of π_m is:

$$\mu \in \mathscr{A}_1'(X), \nu \in \mathscr{A}_1'(Y) : \pi_m{}^*(\mu, \nu) = \mu \otimes \nu \in \mathscr{A}_1'(X \times Y). \tag{I.17}$$

For every $g \in \mathscr{A}(X)$ the function πg is clearly affine with respect to μ and ν, and continuous if we supply $\mathscr{A}_1'(X)$ and $\mathscr{A}_1'(Y)$ with their weak topologies. For these topologies $\mathscr{A}_1'(X)$ and $\mathscr{A}_1'(Y)$ are moreover convex compact. Then $\pi_m g$ has a saddle point (see [2]) and a value. This proves the mixed extension is playable. We denote by $v_m(g)$, the "mixed value" of g.

(b) We give now another example: Let $\mathscr{F}(Y, X)$ be the set of mappings from Y into X. There is a canonical embedding i from X into $\mathscr{F}(Y, X)$

$$i : X \to \mathscr{F}(Y, X)\ \forall y \in Y\ i(x)(y) = x. \tag{I.18}$$

If $P \in \mathscr{F}(Y, X)$ maps Y into X and $y \in Y$, we put:

$$\forall g \in \mathscr{A}(X \times Y)\ \pi g(P, y) = g(P(y), y). \tag{I.19}$$

We have thus defined an extension p:

$$p = (\mathscr{F}(Y, X), Y, i, j, \pi) \tag{I.20}$$

where j is the identity mapping. The meaning of this extension is the following: Yves chooses first his pure strategy y, and then Xavier chooses his pure strategy x *knowing* y. Clearly this extension is playable and the value of πg is $\inf_y \sup_x g(x, y)$. The extension (I.20) is thus the best one for Xavier.

(c) Finally we give a third example of extension. In this extension we have: $X = Y = \{1, 2\}$. A function $g \in \mathscr{A}(X \times Y)$ can be represented (as is usual in game theory) as a (2–2) matrix whose rows "are" the elements of X and whose columns "are" the elements of Y:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{I.21}$$

Now we have: $\mathscr{X} = \mathscr{Y} = \{1, 2, 3, 4\}$ and the one-to-one mappings i and j are simply the identity mapping. The extended payoff function πg can also be described by a (4–4) matrix, and the property (I.7) shows that the upper left (2–2) submatrix of this matrix is g. We consider now the particular example in which the extended payoff function πg of the previous g (I.21) is:

$$\pi g = \begin{bmatrix} a & b & a & b \\ c & d & d & c \\ a & d & a & \left(\dfrac{a+b+c+d}{4}\right) \\ c & b & \left(\dfrac{a+b+c+d}{4}\right) & c \end{bmatrix}. \tag{I.22}$$

One verifies easily that π satisfies properties (I.4) and (I.5). Thus (I.22) describes an extension of the games on $X \times Y$. The "way of playing" the initial game corresponding to this abstract extension is not so intuitive. In fact we shall see in II, Section 3 that this extension (I.22) is the ergodic one, which has been already described in the introduction. But one can note directly in (I.22) that this extension is playable, that is, this matrix (I.22) has a saddle point for every given a, b, c, d (we leave the proof to the reader).

II. Extensions and Exchange of Information

II.1. *Extensions without Exchange of Information*

In this section we assume that X and/or Y is finite.

We identify $\mathscr{A}(X)$ with the subset of $\mathscr{A}(X \times Y)$ of those functions $g(x, y)$ which do not depend on y. Similarly, $\mathscr{A}(Y)$ is regarded as the subset of $\mathscr{A}(X \times Y)$ of those functions which do not depend on x.

Definition II.1. An extension $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ is said to be without exchange of information if the following property holds:

$$\pi[\mathscr{A}(X)] \subset \mathscr{A}(\mathscr{X}) \quad \text{and} \quad \pi[\mathscr{A}(Y)] \subset \mathscr{A}(\mathscr{Y}). \tag{II.1}$$

If the initial payoff function g does not depend on y ($g \in \mathscr{A}(X)$), this means that Yves is a "dummy" in this game. If (II.1) holds, Yves remains a "dummy" in the extended game ($\pi g \in \mathscr{A}(\mathscr{X})$). We claim that, if Yves were no longer a dummy in the extended game ($\pi g \notin \mathscr{A}(\mathscr{X})$), then Yves has some influence on the extended payoff function, and this influence necessary comes from an "exchange of information." For instance, if $p = (\mathscr{F}(Y, X), Y, i, j, \pi)$ is the previous example (I.20), then we have (by (I.19)), if g belongs to $\mathscr{A}(X)$:

$$\pi g(P, y) = g(P(y), y) = g(P(y)). \tag{II.2}$$

Thus, Yves is not a dummy in the extended game (of course if Xavier plays "well," the influence of Yves in the extended game is not relevant).

The typical example of an extension without exchange of information is the mixed one, for which (II.1) clearly holds. We show, in Theorem II.1 below, that all the extensions without exchange of information are closely related to the mixed one. This is another justification of the term "without exchange of information." The sequential extension (see Definition II.2 below) also illustrates this term. But the main justification is Theorem III.2 (part III, Section 2).

THEOREM II.1. *Let $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ be an extension without exchange of information. Then for every initial payoff function $g \in \mathscr{A}(X \times Y)$, if $v_m(g)$ denotes the mixed value of g, we have*:

$$\begin{aligned}\sup_x \inf_y g(x, y) \leqslant \sup_\xi \inf_\eta \pi g(\xi, \eta) \leqslant v_m(g) &\leqslant \inf_\eta \sup_\xi \pi g(\xi, \eta)\\ &\leqslant \inf_y \sup_x g(x, y).\end{aligned} \tag{II.3}$$

Theorem II.1 implies that, if an extension without exchange of information is playable, then the value of the extended game πg is the mixed value of g (for every g). The playable extensions without exchange of information are then "other versions" of the mixed game.

Proof of Theorem II.1. As a consequence of relation (II.1) one shows first that, for every $\xi \in \mathscr{X}$, there exists a linear form $a(\xi) \in \mathscr{A}_1'(X)$ such that:

$$\forall y \in Y \; \pi^*(\xi, j(y)) = a(\xi) \otimes \delta_y . \tag{II.4}$$

Similarly, for every $\eta \in \mathscr{Y}$, there exists a linear form $b(\eta) \in \mathscr{A}_1'(Y)$ such that:

$$\forall x \in X \; \pi^*(i(x), \eta) = \delta_x \otimes b(\eta). \tag{II.5}$$

Now, the relation (II.4) implies, for every fixed ξ:

$$\inf_{\eta \in \mathscr{Y}} \pi g(\xi, \eta) \leqslant \inf_{y \in Y} [\pi^*(\xi, j(y)), g] = \inf_{y \in Y} [a(\xi) \otimes \delta_y , g]. \tag{II.6}$$

Note that

$$\forall \mu \in \mathscr{A}_1'(X) \inf_{y \in Y}[\mu \otimes \delta_y, g] = \inf_{\nu \in \mathscr{A}_1'(Y)}[\mu \otimes \nu, g] \leqslant v_m(g). \qquad \text{(II.7)}$$

Then (II.6) and (II.7) imply the left part of inequalities (II.3). Similarly (II.5) implies the right part. ∎

We now define a family of extensions without exchange of information: The sequential extensions. We denote by $\tilde{X}$ the set of sequences $\tilde{x} = (x_i)_{i \in \mathbb{N}}$ with elements in X, and by $\tilde{Y}$ the set of sequences $\tilde{y} = (y_j)_{j \in \mathbb{N}}$ with elements in Y. There is a natural embedding i from X into $\tilde{X}$, and j from Y into $\tilde{Y}$:

$$\begin{aligned} &i \colon X \to \tilde{X}\ i(x) = (x, \ldots, x, \ldots); \\ &j \colon Y \to \tilde{Y}\ j(y) = (y, \ldots, y, \ldots). \end{aligned} \qquad \text{(II.8)}$$

Definition II.2. Let $a = (a_{ij})_{i,j \in \mathbb{N}}$ be an element of $l^1(\mathbb{N}^2)$ such that:

$$\forall i, j \colon a_{ij} \geqslant 0; \qquad \sum_{i,j=1}^{+\infty} a_{ij} = 1. \qquad \text{(II.9)}$$

Then $p_a = (\tilde{X}, \tilde{Y}, i, j, \pi_a)$ is an extension, where π_a is defined by:

$$\forall (\tilde{x}, \tilde{y}) \in \tilde{X} \times \tilde{Y}, \qquad \pi_a g(\tilde{x}, \tilde{y}) = \sum_{i,j=1}^{+\infty} a_{ij} g(x_i, y_j). \qquad \text{(II.10)}$$

We call it a sequential extension. It is an extension without exchange of information.

A sequential extension is described by the following "way of playing" the initial game: Xavier chooses secretly a sequence $(x_i)_{i \in \mathbb{N}}$ of pure strategies, and Yves chooses secretly a sequence $(y_j)_{j \in \mathbb{N}}$ of pure strategies. The payoff is then the "generalized discounted" summation (II.10) of the $g(x_i, y_j)$.

Theorem II.2. *If X and Y are finite ($|X| = n$, $|Y| = p$) there exists an element $a = (a_{ij})_{i,j \in \mathbb{N}}$ such that the associated sequential extension p_a is playable (and the value of $\pi_a g$ is the mixed value of g). For instance, $p_{\bar{a}}$ is playable with the following $\bar{a}$:*

$$\bar{a} = (\bar{a}_{ij})_{i,j \in N}, \qquad \bar{a}_{ij} = \frac{1}{np}\left(\frac{n-1}{n}\right)^{i-1}\left(\frac{p-1}{p}\right)^{j-1}. \qquad \text{(II.11)}$$

Proof. A standard computation shows that the adjoint form $\pi_{\bar{a}}^*$ of $p_{\bar{a}}$ is:

$$\pi_{\bar{a}}^*(\tilde{x}, \tilde{y}) = \left[\frac{1}{n}\sum_{i=1}^{+\infty}\left(\frac{n-1}{n}\right)^{i-1}\delta_{x_i}\right] \otimes \left[\frac{1}{p}\sum_{j=1}^{+\infty}\left(\frac{p-1}{p}\right)^{j-1}\delta_{y_j}\right]. \qquad \text{(II.12)}$$

The second step is to prove that, if X has n elements, the following mapping is surjective:

$$\alpha: \tilde{X} \to \mathscr{A}_1'(X): \alpha(\tilde{x}) = \frac{1}{n} \sum_{i=1}^{+\infty} \left(\frac{n-1}{n}\right)^{i-1} \delta_{x_i}. \tag{II.13}$$

The end of the proof is very similar to the proof of Theorem II.1: If the adjoint form π^* of an extension can be expressed as:

$$\pi^*(\xi, \eta) = \alpha(\xi) \otimes \beta(\eta), \tag{II.14}$$

where α is onto $\mathscr{A}_1'(X)$ and β onto $\mathscr{A}_1'(Y)$, then this extension is playable. ∎

Thus, we have a way of playing the initial game without moves of chance or randomized strategies, which is nevertheless "equivalent" to the mixed way: Indeed, optimal strategies in the extended game $\pi_{\bar{a}}g$ are well-determined sequences of pure strategies. Moreover, if we restrict the summation (II.10) to the N first terms:

$$\pi^N g(\tilde{x}, \tilde{y}) = \sum_{i,j=1}^{N} a_{ij} g(x_i, y_j), \tag{II.15}$$

we obtain an extended game with a *finite* set of extended strategies. Theorem (II.2) shows that, using the finite N sequences $(x_1 \cdots x_N)$ as extended strategies of Xavier and finite N sequences $(y_1 \cdots y_N)$ as extended strategies of Yves, one obtains a game $\pi^N g$ which is equivalent to the mixed game *up to* $\epsilon(N)$:

$$\epsilon(N) = \| g \| \left(\left(\frac{n-1}{n}\right)^N + \left(\frac{p-1}{p}\right)^N\right). \tag{II.16}$$

(This means that the duality gap of $\pi^N g$ is less than $2\epsilon(N)$ and contains the mixed value, and that there exist $\epsilon(N)$-optimal strategies for both players.)

In the case where X or Y, but not both, is infinite it is possible, by the Hahn–Banach theorem, to obtain sequential extensions which are equivalent to the mixed extension (see [4]).

Remark II.1. Aumann's "randomized and correlated strategies" (see [1]) are closely related to the sequential extension. Indeed, it can be shown that, if the lottery is countable, there is a complete isomorphism from the correlated strategies (with a single "objective" probability) onto the sequential extensions (Definition II.2). For more details see [4].

II.2. *Extensions by Pure Exchange of Information*

We denote by $\mathscr{F}(Y, X)$ (and $\mathscr{F}(X, Y)$) the set of mappings from Y into X (and from X into Y). The canonical embedding j from Y into $\mathscr{F}(X, Y)$ is:

$$j : Y \to \mathscr{F}(X, Y) \quad \forall x \in X j(y)(x) = y. \tag{II.17}$$

Similarly we denote by i the canonical embedding from X into $\mathscr{F}(Y, X)$ (see (I.18)). If ξ belongs to $\mathscr{F}(Y, X)$ and η belongs to $\mathscr{F}(X, Y)$ we denote by $F(\xi, \eta)$ the subset (which may be empty) of the fixed points of the mapping: $(x, y) \to (\xi(y), \eta(x))$. That is:

$$F(\xi, \eta) = \{(x, y) \in X \times Y / \xi(y) = x \quad \text{and} \quad \eta(x) = y\}. \tag{II.18}$$

DEFINITION II.3. A pair $(\mathscr{X}, \mathscr{Y})$ such that:

$$X \subset \mathscr{X} \subset \mathscr{F}(Y, X), \tag{II.19}$$

$$Y \subset \mathscr{Y} \subset \mathscr{F}(X, Y), \tag{II.20}$$

is called a feasible pair if the following holds:

$$\forall (\xi, \eta) \in \mathscr{X} \times \mathscr{Y} \; F(\xi, \eta) \neq \varnothing. \tag{II.21}$$

DEFINITION II.4. Let $(\mathscr{X}, \mathscr{Y})$ be a feasible pair. Let us choose for every pair $(\xi, \eta) \in \mathscr{X} \times \mathscr{Y}$ an element $f(\xi, \eta)$ of $F(\xi, \eta)$. Define the operator π by:

$$\forall g \in \mathscr{A}(X \times Y) \quad \forall (\xi, \eta) \in \mathscr{X} \times \mathscr{Y} \quad \pi g(\xi, \eta) = g(f(\xi, \eta)). \tag{II.22}$$

Then $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ is an extension of the games on $X \times Y$ and we call p an extension by pure exchange of information.

The extended payoff function πg has the same range as g, but using their extended strategies, which are in fact "decision rules," both players use some information about the other player's behavior. For instance, if $(\mathscr{X}, \mathscr{Y})$ is the feasible pair $(X, \mathscr{F}(X, Y))$, the only associated extension by pure exchange of information is given by:

$$\forall g \in \mathscr{A}(X \times Y) \quad \pi g(x, \eta) = g(x, \eta(x)). \tag{II.23}$$

In this extension, Yves has full information about Xavier's behaviour. In other words, Xavier chooses first his pure strategy and then Yves, knowing this choice, chooses his pure strategy. This extension is playable, with value $\sup_x \inf_y g(x, y)$ and is the opposite of the previous Example I.3(b), in which Xavier has full information. Note that the pair $(\mathscr{F}(Y, X), \mathscr{F}(X, Y))$ is not feasible: Both players can not have full information about the other player's behavior!

We now describe a more complex exchange of information, which leads us to another example of extension by pure exchange of information. Let r be an equivalence relation on X and s be an equivalence relation on Y. If x is a pure strategy of Xavier, $r(x)$ denotes its equivalence class in the quotient X/r. Similarly $s(y)$ denotes the equivalence class of y in Y/s. Knowing $r(x)$, Yves has partial information about Xavier's behaviour. We consider the following scheme of exchange of information: First Xavier chooses his $r(x)$ (but not x) and then Yves, knowing Xavier's choice, chooses his $s(y)$ (but not y). Then Xavier, knowing Yves's choice, chooses x according to his previous promise (that is $x \in r(x)$) and finally Yves, knowing Xavier's pure strategy x, chooses his pure strategy y, according to his previous promise (that is $y \in s(y)$). One verifies easily that this scheme is described by the feasible pair $(\mathscr{X}, \mathscr{Y})$ where:

$$\mathscr{X} = \{\xi \in \mathscr{F}(Y, X)/s(y) = s(y') \Rightarrow \xi(y) = \xi(y') \text{ and } r(\xi(y)) \text{ does not depend on } y\} \tag{II.24}$$

and

$$\mathscr{Y} = \{\eta \in \mathscr{F}(X, Y)/r(x) = r(x') \Rightarrow s(\eta(x)) = s(\eta(x'))\}. \tag{II.25}$$

Moreover there is only one extension by pure exchange of information associated with this feasible pair because $F(\xi, \eta)$ contains only one element, and this extension is playable. This example can be generalized to a scheme of alternate exchanges of partial information (see [4]).

The next result (Theorem (II.3) below) is a characterization of playable extensions by pure exchange of information and gives a precise meaning to the intuitive feeling: These extensions are playable if the exchange of information is "maximal."

DEFINITION II.5. Let A be a family of subsets of $X \times Y$. We call conjugate of A and denote by A^* the following set of subsets of $X \times Y$:

$$A^* = \{T \subset X \times Y / \forall S \in A \; T \cap S \neq \varnothing\}. \tag{II.26}$$

LEMMA II.1. *We have*

$$A^{**} = \{T \subset X \times Y / \exists S \in A \; S \subset T\}. \tag{II.27}$$

In the next theorem we identify a mapping $\xi \in \mathscr{F}(Y, X)$ and/or a mapping $\eta \in \mathscr{F}(X, Y)$ with their graphs in $X \times Y$.

THEOREM II.3. *Let $(\mathscr{X}, \mathscr{Y})$ be a pair such that*

$$X \subset \mathscr{X} \subset \mathscr{F}(Y, X) \tag{II.28}$$

$$Y \subset \mathscr{Y} \subset \mathscr{F}(X, Y). \tag{II.29}$$

Then the four following properties are equivalent

(i) $(\mathscr{X}, \mathscr{Y})$ *is a feasible pair and one of the associated extensions by pure exchange of information is playable*;

(ii) $(\mathscr{X}, \mathscr{Y})$ *is a feasible pair and every associated extension by pure exchange of information is playable* (*with the same value*);

(iii) $\mathscr{X}^{**} = \mathscr{Y}^*$;

(iv) $\mathscr{Y}^{**} = \mathscr{X}^*$.

Property (iii) implies that, given $\mathscr{Y}$, every decision rule of Xavier $\xi \in \mathscr{F}(Y, X)$ which is "feasible with respect to $\mathscr{Y}$" ($\xi \in \mathscr{Y}^*$) belongs to $\mathscr{X}$. Then $\mathscr{X}$ is exactly the set of those decision rules of Xavier which are feasible with respect to $\mathscr{Y}$. Similarly $\mathscr{Y}$ is the set of those decision rules of Yves which are feasible with respect to $\mathscr{X}$. In the case of finite X and Y one can prove a converse of this property ("if $\mathscr{X}$ is the set of feasible decision rules with respect to $\mathscr{Y}$, then the extensions associated with $(\mathscr{X}, \mathscr{Y})$ are playable"). For more details see [4].

Proof of Theorem II.3. Let us assume that (i) holds. We want to prove that (iii) holds. By feasibility $\mathscr{X} \subset \mathscr{Y}^*$, and then $\mathscr{X}^{**} \subset \mathscr{Y}^*$ (use Lemma (II.1)). In order to prove $\mathscr{Y}^* \subset \mathscr{X}^{**}$, suppose that there exists $A \in \mathscr{Y}^*$ and $A \notin \mathscr{X}^{**}$. Then, using Lemma II.1 we have:

$$\forall \xi \in \mathscr{X} \ \xi \not\subset A \tag{II.30}$$

$$\forall \eta \in \mathscr{Y} \ \eta \cap A \neq \varnothing. \tag{II.31}$$

The relations (II.30) and (II.31) can be expressed as:

$$\forall \xi \ \exists y (\xi(y), y) \notin A \tag{II.32}$$

$$\forall \eta \ \exists x (x, \eta(x)) \in A. \tag{II.33}$$

Define $g \in \mathscr{A}(X \times Y)$ by:

$$g(x, y) = 1 \quad \text{if} \quad (x, y) \in A; \ g(x, y) = 0 \quad \text{if} \quad (x, y) \notin A. \tag{II.34}$$

Then, clearly, if π is the operator of an extension by pure exchange of information associated with $(\mathscr{X}, \mathscr{Y})$, the function πg has a duality interval equal to $[0, 1]$. This contradicts (i), and (iii) is proved. The end of the proof is to establish that property (iii) implies property (ii). One uses typically the same kind of argument. ∎

II.3. *Compound Extensions. The Iterated Games*

In this section we describe, in terms of extensions, the "partial exchange of information." For that purpose, we shall compound an extension without exchange of information with an extension by pure exchange of information. In the next part (III, Section 2) we shall show that "all" the extensions are equivalent (in a certain sense) to such extensions.

PROPOSITION II.1. *Let* $p_1 = (\mathscr{X}_1, \mathscr{Y}_1, i_1, j_1, \pi_1)$ *be an extension of the games on* $X \times Y$ *and* $p_2 = (\mathscr{X}_2, \mathscr{Y}_2, i_2, j_2, \pi_2)$ *be an extension of the games on* $\mathscr{X}_1 \times \mathscr{Y}_1$ (π_2 *maps* $\mathscr{A}(\mathscr{X}_1 \times \mathscr{Y}_1)$ *into* $\mathscr{A}(\mathscr{X}_2 \times \mathscr{Y}_2)$). *Then* $p = (\mathscr{X}_2, \mathscr{Y}_2, i_2 \circ i_1, j_2 \circ j_1, \pi_2 \circ \pi_1)$ *is an extension of the games on* $X \times Y$. *We call* p *the extension "*p_1 *compounded with* p_2*."*

The proof is standard.

DEFINITION II.6. An extension with partial exchange of information is an extension without exchange of information compounded with an extension by pure exchange of information.

We just illustrate this definition by the example of iterated games. Consider a sequential extension (Definition (II.2)) p_a. The extended strategy spaces of such an extension are $\tilde{X}$ and $\tilde{Y}$. Now, consider an extension by pure exchange of information p of the games on $\tilde{X} \times \tilde{Y}$: We only describe the scheme of exchange of information associated with p (for a precise formulation, see [4]). Let A and B be two subsets of $\mathbb{N}$. The scheme is as follows: Xavier chooses x_1; Yves knows x_1 if $1 \in B$ (otherwise he does not know x_1). Yves chooses y_1; Xavier knows y_1 if $1 \in A$ (otherwise he does not know y_1). Xavier chooses x_2; Yves knows x_2 if $2 \in B$; Yves chooses y_2 and so on. Then the set A is the set of those i such that, before he chooses x_{i+1}, Xavier knows y_i. The set B is the set of those j such that, before he chooses y_j, Yves knows x_j. If $A = B = \mathbb{N}$, then the information is "perfect," and if $A = B = \varnothing$, there is no exchange of information.

The compound of p_a with p is called an iterated game, and is an extension with partial exchange of information. In general, an iterated game is not playable, but one can show that the duality gap of the iterated game described before is bounded by:

$$2 \| g \| \cdot \left(1 - \sum_{\substack{i \in A \\ i \leqslant j}} a_{ij} - \sum_{\substack{j \in B \\ j < i}} a_{ij} \right) \tag{II.35}$$

(where g is the initial payoff function). This bound illustrates the intuitive feeling: The larger is the exchange of information, the less is the duality gap.

A specially interesting iterated game is the *ergodic game* described in the Introduction (see (0.7)). In fact, it is a limiting case of a sequential game and its definition needs the Hahn–Banach theorem. Indeed, the limit

$$\lim_{N \to +\infty} \frac{1}{2N} \sum_{i=1}^{N} [g(x_i, y_i) + g(x_{i+1}, y_i)] \tag{II.36}$$

does not always exist. However, the Hahn–Banach theorem asserts that we can extend the "Cesaro mean limit" operator to all sequences of real numbers, and then the ergodic game is well defined as an extension with partial exchange of information. The main result is that the ergodic game is playable, that there are stationary optimal strategies, and that neither the value, nor the optimal strategies depend on the particular extension we have chosen for the Cesaro mean limit operator. This result is well known in the case of finite X and Y (see [3]) and follows from [5] in the case of compact X and Y with continuous g.

III. Value Functions

III.1. *Characterization of Value Functions*

Let $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ be a *playable* extension of the games on $X \times Y$. We call the value function associated with p, denoted v_p, the function

$$v_p\colon \mathscr{A}(X \times Y) \to \mathbb{R},$$
$$v_p(g) = \sup_{\xi} \inf_{\eta} \pi g(\xi, \eta) = \inf_{\eta} \sup_{\xi} \pi g(\xi, \eta). \tag{III.1}$$

Definition III.1. A value function is a function v

$$v\colon \mathscr{A}(X \times Y) \to \mathbb{R} \tag{III.2}$$

such that there exists a playable extension p of the games on $X \times Y$ satisfying:

$$\forall g \in \mathscr{A}(X \times Y) \quad v(g) = v_p(g). \tag{III.3}$$

If we consider as *equivalent* two playable extensions with the same value function, the next result is then a characterization of the playable extensions.

Theorem III.1. *A function $v\colon \mathscr{A}(X \times Y) \to \mathbb{R}$ is a value function if and only if the four following properties hold*:

$$\forall g\, \forall \lambda \geqslant 0\ v(\lambda g) = \lambda v(g), \tag{III.4}$$

$$\forall g\, \forall \lambda \qquad v(g + \lambda\theta) = v(g) + \lambda, \tag{III.5}$$

$$\forall g_1, g_2 (g_1 \geqslant g_2) \Rightarrow (v(g_1) \geqslant v(g_2)), \tag{III.6}$$

$$\forall g\ \sup_x \inf_y g(x, y) \leqslant v(g) \leqslant \inf_y \sup_x g(x, y). \tag{III.7}$$

Properties (III.4) and (III.5) mean that v is "affine" for very simple affine transformations of the payoff g. Property (III.6) means that v is increasing with respect to the partial ordering of $\mathscr{A}(X \times Y)$ and property (III.7) means that the "value" always belongs to the duality interval. These four very "natural" properties are an "a posteriori" justification to the basic assumption of *linearity* of the extension operator. Indeed, in Definition I.1 this assumption seems to be there for mathematical convenience only. But Theorem III.1 shows that this leads us to all "possible" value functions. What could be the interest of "nonlinear" extensions, leading to some "value function" for which one of the properties (III.4) to (III.7) fails?

Remark III.1. In Theorem III.1 we can replace properties (III.5) and (III.6) by the equivalent (III.8):

$$\forall g_1, g_2, \qquad v(g_1) - v(g_2) \leqslant \sup_{(x,y)\in X\times Y} (g_1(x, y) - g_2(x, y)). \qquad \text{(III.8)}$$

We could express property (III.8) as: v is "positively 1-Lipschitzian."

Proof of Theorem III.1. First of all the definition of an extension and Proposition I.1 clearly imply that, if v is a value function, then v verifies (III.4) to (III.7). The difficult part of the proof is the converse.

Let V be the set of value functions.

Step 1. *If* $(v_\alpha)_{\alpha\in A}$ *is a family of elements of* V, *then* $\sup_{\alpha\in A} v_\alpha$ *and* $\inf_{\alpha\in A} v_\alpha$ *belong to* V.

Let $p_\alpha = (\mathscr{X}_\alpha, \mathscr{Y}_\alpha, i_\alpha, j_\alpha, \pi_\alpha)$ be a playable extension with value function v_α. We just consider the case where $\alpha = 1, 2$. The general case is no more difficult. Let $\mathscr{X}$ be the "sum" of $\mathscr{X}_1$ and $\mathscr{X}_2$ ($\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2$ if $\mathscr{X}_1$ and $\mathscr{X}_2$ are disjoint). Let $\mathscr{Y} = \mathscr{Y}_1 \times \mathscr{Y}_2$ and j be the one-to-one mapping:

$$j\colon Y \to \mathscr{Y} \quad j(y) = (j_1(y), j_2(y)). \qquad \text{(III.9)}$$

Define now the linear operator π by its adjoint form π^* (see Proposition I.2)

$$\forall(\xi, (\eta_1, \eta_2)) \in \mathscr{X} \times \mathscr{Y} \qquad \pi^*(\xi, (\eta_1, \eta_2)) = \pi^*(\xi, \eta_1) \quad \text{if} \quad \xi \in \mathscr{X}_1$$
$$= \pi^*(\xi, \eta_2) \quad \text{if} \quad \xi \in \mathscr{X}_2. \qquad \text{(III.10)}$$

Then $p = (\mathscr{X}, \mathscr{Y}, i_1, j, \pi)$ is an extension of the games on $X \times Y$ and one verifies that p is playable with value $\sup(v_1, v_2)$.

Step 2. *Let* $v_0 : \mathscr{A}(X \times Y) \to \mathbb{R}$ *be a function such that*

$$\forall g_1, g_2 \in \mathscr{A}(X \times Y)\colon \exists v \in V\ v_0(g_1) = v(g_1) \quad \text{and} \quad v_0(g_2) = v(g_2). \qquad \text{(III.11)}$$

Then v_0 *belongs to* V.

Put

$$w = \sup\{v/v \in V, v \leqslant v_0\} \leqslant v_0. \qquad \text{(III.12)}$$

By assumption (III.7), w is well defined and by Step 1, w belongs to V. For any fixed $g_0 \in \mathscr{A}(X \times Y)$, put

$$v_{g_0} = \inf\{v/v \in V, v(g_0) = v_0(g_0)\}. \tag{III.13}$$

By assumption (III.11) v_{g_0} is well defined and, moreover

$$v_{g_0} \leqslant v_0 . \tag{III.14}$$

By Step 1, v_{g_0} belongs to V, and then (III.14) implies:

$$v_{g_0} \leqslant w \leqslant v_0 . \tag{III.15}$$

In particular

$$v_{g_0}(g_0) = v_0(g_0) \leqslant w(g_0) \leqslant v_0(g_0). \tag{III.16}$$

This proves $v_0 = w$ and Step 2 is proved.

Step 3. *Let* v: $\mathscr{A}(X \times Y) \to \mathbb{R}$ *be a function such that* (III.4) *to* (III.7) *hold. Then for every fixed* g_1 *and* g_2 *in* $\mathscr{A}(X \times Y)$ *there exists an element* m *of* $\mathscr{A}_1'(X \times Y)$ *such that*:

$$[m, g_1] \leqslant v(g_1) \qquad \textit{and} \qquad v(g_2) \leqslant [m, g_2]. \tag{III.17}$$

Consider the following two person zero sum game: The strategy set of player 1 is $\{1, 2\}$ and the strategy set of the second player is $X \times Y$. The payoff function is L:

$$\begin{aligned} L(1, (x, y)) &= g_1(x, y) - v(g_1), \\ L(2, (x, y)) &= -g_2(x, y) + v(g_2). \end{aligned} \tag{III.18}$$

Let α be the *mixed* value of this game. One proves first, using the mixed strategies of player 1, and properties (III.4) and (III.8), that α is less than or equal to 0. One verifies then, that an optimal strategy m of player 2 satisfies inequality (III.17).

Step 4. *End of the proof.* Let v: $\mathscr{A}(X \times Y) \to \mathbb{R}$ be a function such that (III.4) to (III.7) hold. By Step 3, for every fixed g_1 and g_2 in $\mathscr{A}(X \times Y)$ there exist m_1 and m_2 in $\mathscr{A}_1'(X \times Y)$ such that:

$$\begin{aligned} [m_1 , g_1] \leqslant v(g_1) \leqslant [m_2 , g_1] \\ [m_2 , g_2] \leqslant v(g_2) \leqslant [m_1 , g_2]. \end{aligned} \tag{III.19}$$

One constructs then an extension p, for which each extended strategy set has exactly two nonpure extended strategies, and which verifies:

$$v(g_i) = \sup_{\xi} \inf_{\eta} \pi g_i(\xi, \eta) \qquad i = 1, 2. \tag{III.20}$$

From (III.20) one deduces easily that there exists $\bar{v} \in V$ such that

$$v(g_i) = \bar{v}(g_i) \qquad i = 1, 2. \tag{III.21}$$

Finally, v satisfies (III.11) and then belongs to V. ∎

III.2. *Value Functions and Compound Extensions*

Let us say that two playable extensions are *equivalent* if they have the same value function.

THEOREM III.2. *Every playable extension is equivalent to an extension with partial exchange of information. That is, if v is a value function, there exist an extension without exchange of information and an extension by pure exchange of information, whose compounded p satisfies*:

$$\forall g \in \mathscr{A}(X \times Y), \qquad v(g) = \sup_{\xi} \inf_{\eta} \pi g(\xi, \eta) = \inf_{\eta} \sup_{\xi} \pi g(\xi, \eta). \tag{III.22}$$

Proof. *Step* 1. Let $(p_\alpha)_{\alpha \in A}$ be a family of extensions with partial exchange of information: For every α, p_α is the compound of $p_\alpha{}^1$, without exchange of information, with $p_\alpha{}^2$, by pure exchange of information. We assume that, for every α, $p_\alpha{}^1$ verifies:

$$\forall y (\pi_\alpha{}^1)^* \, (\mathscr{X}_\alpha{}^1 \times \{j_\alpha(y)\}) = \mathscr{A}_1'(X) \otimes \delta_y \tag{III.23}$$

and

$$\forall x (\pi_\alpha{}^1)^* \, (\{i_\alpha(x)\} \times \mathscr{Y}_\alpha{}^1) = \delta_x \otimes \mathscr{A}_1'(Y). \tag{III.24}$$

Now, using the same kind of construction as in Step 1 of the proof of Theorem III.1, one can prove that there exists an extension p with partial exchange of information (p is the compound of p_1 with p_2) such that p_1 verifies (III.23) and (III.24), and such that p is playable with value function $\sup_{\alpha \in A} (v_\alpha)$ (where v_α is the value function of p_α). The same result is true with the infimum instead of the supremum.

Step 2. Let m be an element of $\mathscr{A}_1'(X \times Y)$. Define v_m by:

$$\forall g \in \mathscr{A}(X \times Y),$$

$$v_m(g) = \begin{cases} \sup_x \inf_y g(x, y) & \text{if} \quad [m, g] \leqslant \sup_x \inf_y g(x, y) \\ \inf_y \sup_x g(x, y) & \text{if} \quad \inf_y \sup_x g(x, y) \leqslant [m, g] \\ [m, g] & \text{otherwise.} \end{cases} \tag{III.25}$$

One proves that v_m is the value function of an extension p with partial exchange of information (p is the compound of p_1 with p_2) such that p_1 verifies (III.23) and (III.24).

Step 3. If v is the value function of an extension $p = (\mathscr{X}, \mathscr{Y}, i, j, \pi)$ we put:

$$\pi^*(\xi, \eta) = m_{\xi,\eta}. \tag{III.26}$$

Then v can be written as

$$\forall g \in \mathscr{A}(X \times Y), \qquad v(g) = \sup_{\xi} \inf_{\eta} [m_{\xi,\eta}, g]. \tag{III.27}$$

A simple computation shows that

$$\forall g \in \mathscr{A}(X \times Y), \qquad v(g) = \sup_{\xi} \inf_{\eta} v_{m_{\xi,\eta}}(g). \tag{III.28}$$

By Step 1 and Step 2 the function v_ξ

$$\forall g \in \mathscr{A}(X \times Y), \qquad v_\xi(g) = \inf_{\eta} v_{m_{\xi,\eta}}(g), \tag{III.29}$$

is then the value function of an extension with partial exchange of information verifying (III.23) and (III.24). Applying Step 1 once more, v is the value function of an extension with partial exchange of information (verifying (III.23) and (III.24)). ∎

References

1. R. J. Aumann, Subjectivity and correlation in randomized strategies, *J. Math. Econ.* 1 (1974), 67–96.
2. K. Fan, A minimax inequality and applications, *in* "Inequalities III. Shista," pp. 103–113, Academic Press, New York, 1972.
3. D. Gillette, Stochastic games with zero stop probability, *in* "Contributions to the Theory of Games," Vol. 3, Princeton University Press, 1957.
4. H. Moulin, Prolongements des jeux à deux joueurs de somme nulle, Thesis, Paris 1975.
5. H. Moulin, Iterated games, *Internat. J. of Game Theory*, to appear.